Dirichlet Surface Isolines

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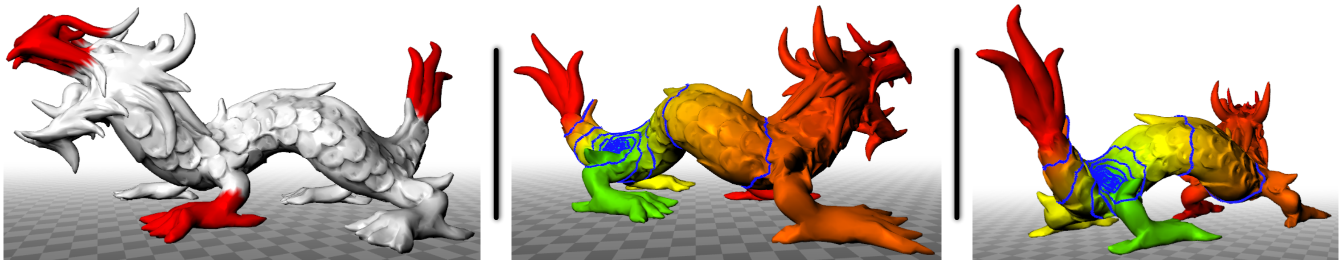

Figure 1: ***Competitive-boundaries for surface isolines.*** *Competitive boundaries are specified by the user (in red color). A brush-based tool allows us to paint multiple topological disks acting as boundaries defined over the surface (**left-hand side**). Then, a color-coded harmonic field is computed by solving the boundary-aware Poisson equation (**middle**). Finally, isolines (superimposed in blue lines) are extracted by sampling the harmonic field at given value (**right-hand side**).*

Abstract

Harmonic isolines have proven to be a useful tool for shape analysis. However, the Dirichlet problem for the Poisson equation is still a challenging numerical problem, since boundary conditions are imposed and fixed. In this work, we explore the use of an injective scheme, as an alternative technique to enforce the boundary condition firmly. Our method preserves a square matrix form treatment. At the core of our system lies a linear numerical solver that accurately estimates a harmonic field. The concept of Dirichlet problem is closely related to isoline construction. Our key idea is to formulate the boundary problem as strong constraints inside the Laplacian-type framework. In this regard, our insight is to exploit the linear algebra mechanism to inject the boundary condition directly into the numerical system by wrapping the Laplacian matrix. Our results demonstrate the efficiency and stability of our system in solving a harmonic field and in extracting corresponding isolines. Finally, we show the practicability of our problem by sampling Poisson mesh isolines.

1 Introduction

"Feel the power of the curves."
— Miguel Chevalier

The exploration of intrinsic relationships between curves and surface is a fundamental task for aesthetic visualization of complex objects. Above all, our work is inspired by the visual fluctuations of intricately colored patterns offered by the "Origin of the Curve" digital artwork of Miguel Chevalier. In this paper, we are particularly interested to provide a high-performance visualization for a complex surface. To achieve such goal, we devise an elegant curvy design coupled-and-guided by gradient coloring.

Diffusive Visual Computing. Nowadays, diffusion-based schemes are extensively used in shape analysis. Numerous applications take advantage of smooth and harmonic properties of propagation scheme. In this work, we are interested in studying boundary value problems for Poisson equation. In particular, we focus on Dirichlet boundary condition for Laplacian-type systems. Our main motivation is to provide a closed-form, squared matrix form and a clear understanding of the well-known expression *"subject to boundary condition"*. To this end, our contribution is a novel mathematical formulation called *Injective Dirichlet Condition*. Therefore, we introduce the concept of competitive-boundaries for surface isolines, allowing non-professional users to control the harmonic field and to obtain surface isolines that reflect specified boundaries harmonic influence over the surface.

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The Dirichlet Boundary Problem. The well-known Dirichlet problem is about finding a solution φ for Laplace's equation on some domain Ω such that a given function on the boundary of Ω is equal to a particular function. First of all, a Dirichlet boundary condition for partial differential equations is formalized as a boundary value problem. Then, the equation solution should satisfy the uniqueness of boundary conditions over disjoint regions of the specified boundary domain. When the boundary value is imposed on a particular area of interest, this is known as a Dirichlet boundary condition. Dirichlet boundary condition is also referred to as a fixed boundary condition. The Dirichlet boundary condition is a first-type boundary condition where values are firmly imposed on a Partial Differential Equations. In our work, we focus on Dirichlet boundary conditions for Poisson equations. For our problem of interest, we only consider linear boundary conditions, expressing a linear relation between a function and its partial derivatives. The novelty of our work is an injective scheme tailored for the strong formulation of the Dirichlet boundary condition.

Goal and Motivation. Our key observation is that isolines play a significant role in shape analysis, acting as a high-level descriptor and a surface-based skeleton feature. Our work is motivated by the recent need of isolines to serve as a reduced domain for shape segmentation and mesh deformation. Our work falls into the fundamental problem domain of expressing boundary conditions in a variational fashion. In particular, the central problem of our approach is to determine a matrix concatenation leading to a natural injection of the boundary condition while keeping the numerical system in its square matrix-form. We believe our most impacting contribution is the injection of the boundary constraints into the Laplacian matrix. For convenience, we cast the problem as a variational surface analysis problem. Our main contribution is to rethink the treatment of the boundary condition by avoiding the traditional weak formulation. This flexibility enables a rich set of isolines.

Our Contributions. Our method differs considerably from prior work in the enforcement of the boundary conditions. Even if linear system solvers for mesh processing is a well-explored area [Botsch et al. 2005], the majority of previous techniques are based on a weak formulation that introduces penalty terms as boundary integrals. In contrast, our contribution is based on a strong formulation by overwriting degrees of freedom with the desired value. Our alternative is to employ an injective closed-form scheme. Intuitively, if a weak formulation can be thought of as being valid on average (as seen in Equation 1 of [Au et al. 2011]), a strong formulation would be right almost everywhere.

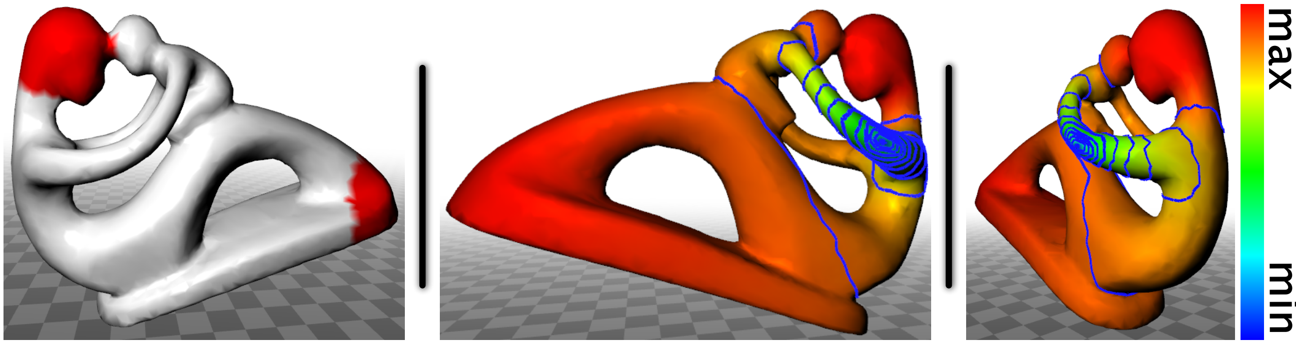

Figure 2: ***Competitive-boundaries for surface isolines.*** *Competitive boundaries are specified by the user (in red color) with a brush-based tool that allows to paint multiple boundaries over the surface (**left-hand side**). Then, a color-coded harmonic field is obtained by solving the boundary-aware Poisson equation (**middle**). Finally, isolines (superimposed in blue lines) are extracted by sampling the harmonic field at given values. The color-code corresponds to the isovalue* $\in [0, 1]$ *of the harmonic field (**right-hand side**).*

2 Related Works

Curves from Surfaces. Curves and surfaces are closely linked by the concept of surface curvature. Curves from surfaces offer a natural structured subspace for abstracting shape representation [De Goes et al. 2011] and shape manipulation [Gal et al. 2009]. The problem of extracting one-dimensional subspace in the form of geometry curves was previously explored by Li *et al.* [2013]. Various techniques have exploited salient surface characteristics to extract curvy feature lines on surface meshes, using discrete Morse theory [Sahner et al. 2008], ridge analysis [Süand Greiner 2007] or discrete differential geometry [Hildebrandt et al. 2005]. In more details, Ohtake *et al.* [2004] propose a scale-independent ridge-valley lines on meshes, defined as curves on a surface. Also, Zou *et al.* [2015] employ input cross-sections curves. The major limitation of such approaches is that resulting lines are not guaranteed to be closed. Even if our work is also relevant to differential geometry of surfaces, our approach is focused on closed curves, namely *isolines* that are agnostics to discontinuous local shape variability like sharp features.

Harmonic Fields. The problem of defining harmonic fields with boundary conditions in a large number of studies. Our work builds upon a rapidly growing body of recent literature on dynamic harmonic fields [Xu et al. 2009], handle-aware isolines [Au et al. 2007] and harmonic volumetric mapping [Li et al. 2007]. More importantly, the majority of the previous methods rely on harmonic isolines for mesh segmentation [Zheng et al. 2011; Zheng and Tai 2010]. For instance, Au *et al.* [2011] have developed mesh segmentation with concavity-aware harmonic fields. Meanwhile, harmonic fields are an essential ingredient to control and segment surfaces. Zheng *et al.* [2010] preferred to decompose the surface with cross-boundary brushes using a simple harmonic field as decomposition support. Our work is also relevant to rearmost efforts in pairwise harmonics [Zheng et al. 2013].

Boundary-Type Methods. Numerical solution approaches for boundary-value problems can be separated into boundary-type methods or domain-type methods [Brebbia et al. 1984]. In Computer Graphics, a numerous of geometric problems are solved by finding solutions simultaneously satisfying differential equations and given boundary conditions. Our work is similar to investigations on boundary condition of Nealen *et al.* [2007] and Jacobson *et al.* [2010] but we address the different problem of Dirichlet for Laplacian-type system [Sorkine et al. 2004].

Isolines. Isoline-based visualization has been extensively used in various applications to highlight how a vector or scalar field flows over the surface. For instance, Xu *et al.* [2014] propose an algorithm for constructing the geodesic Voronoi diagram with polyline generators where iso-distance contours indicate the distance to the generators. Isolines can be seen as characteristic curves reflecting the torsion distribution of a given field. In the context of character articulation, isolines are also employed to visualize deformations imposed by shape-aware weight functions in the vicinity of joints [Jacobson et al. 2012; Kavan and Sorkine 2012].

3 Poisson-Dirichlet Mesh Isolines

In the next section, we detail technical ingredients of our injective scheme. First, we introduce the concept of *Competitive-Boundaries Brushing*. Second, we introduce a novel formulation of the Dirichlet problem and *Injective Dirichlet Condition* techniques resulting in the extraction of a collection of *Dirichlet-Aware Mesh Isolines*.

Competitive-Boundaries Brushing. First, a mesh $\mathcal{M}$ is described by a triplet $\mathcal{M} = (\mathcal{V}; \mathcal{E}; \mathcal{F})$, where $\mathcal{V} = \{v_1, \cdots, v_n\}$ are the set of n vertex position in $\mathbb{R}^3$, $\mathcal{E}$ are the edges that connect two vertices, $\mathcal{F}$ are the faces which connect the edges. Second, we have designed a brush-based interface technique to define the multiple Dirichlet boundary condition. We use a brush-based tool similar to a paintbrush to specify boundaries over the surface. The Dirichlet boundary $\Gamma_D = \{\Pi_0, \cdots, \Pi_l\}$ is defined as a set of l mesh patches, where Π_k is a painted and disjointed region of interest over the surface. Each brushed region of interest ensures C_1 continuity of the surface patch. Each boundary is surface-attached signifying that each discrete boundary is exclusively defined on the surface domain Ω as a genus-zeros collection of connected vertices on the surface. In our work, a typical boundary is represented by a disc topology. Second, all boundaries are competitively fused into our following numerical system to generate a controllable harmonic field.

Boundary-Constrained Harmonics. We assume the input mesh is pre-processed to overcome irregularity in its connectivity. Since the Dirichlet problem can be resolved for many PDEs, our problem is well-posed for Laplace's equation. The Poisson equation is the simplest example of a canonical elliptic partial differential equation. For a domain $\Omega \in R^{3\times n}$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, the Poisson equation with Dirichlet boundary conditions is formulated as follows:

$$\nabla^2\varphi = 0 \quad , \quad \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x} + \frac{\partial^2\varphi}{\partial y} + \frac{\partial^2\varphi}{\partial z} \tag{1}$$

subject to Dirichlet boundary condition:

$$\varphi\big|_{\partial\Omega} = 1 \;\text{ on }\; \Gamma_D \tag{2}$$

where $\varphi : \mathcal{M} \mapsto [0, 1]$ is an unknown scalar function. We denote by ∇^2 the Laplacian operator defined over the geometry surface. The

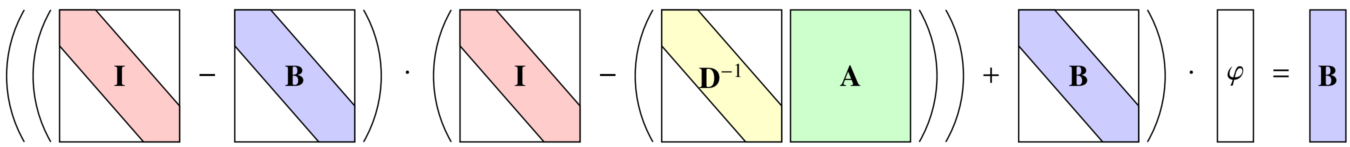

Figure 3: ***Matrix-form Injective Dirichlet Boundary Condition.*** *We present a visual representation of the matrix-form derivation for our injective Dirichlet boundary condition formulation (see Equation 3).*

Laplace operator is the sum of second partial derivative representing the divergence of the function gradient. The desired field φ is guaranteed to be a discrete harmonic field with smoothness properties. Boundary conditions are imposed at different values of the independent variables.

Injective Dirichlet Condition. Poisson's equation is a fundamental model of an elliptic problem and accepts a variational formulation. Hence, the conceivable variations are the functions vanishing on its boundary. Consequently, we cast the problem of Poisson equation with Dirichlet boundary condition as a well-known variational problem. Next, we want to compute a corresponding harmonic scalar field in a closed-form manner such that:

$$\left[(\mathbf{I} - diag\,(\mathbf{B})) \cdot \left(\mathbf{I} - \left(\mathbf{D}^{-1} \cdot \mathbf{A}\right)\right) + diag\,(\mathbf{B})\right] \cdot \varphi = \mathbf{B} \tag{3}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{B}$ is the column matrix storing the Dirichlet boundary condition, $\mathbf{A}$ as the $n \times n$ adjacency matrix, $diag\,(.)$ is the diagonal operator transforming a column matrix into the diagonal matrix, and $\mathbf{D}$ is the diagonal degree matrix. Next, we detail each component with more information. The boundary condition $\varphi\big|_{\partial\Omega}$ is stacked as a column matrix $\mathbf{B}$ with per-vertex rows, such as:

$$\mathbf{B}_i = \begin{cases} 1 & \text{if } \; v_i \in \Gamma_D \\ 0 & \text{if } \; v_i \in \Gamma_N \end{cases} \tag{4}$$

We also note the diagonal degree matrix $\mathbf{D}$, storing the valence of each mesh vertices (i.e. the number of vertex of the one-ring neighborhood $\mathcal{N}_i$) by the following expression:

$$\mathbf{D}_{ii} = |\mathcal{N}_i| \quad \text{with} \quad \mathcal{N}_i = \left\{v_j \;,\; \left(v_i, v_j\right) \in \mathcal{E}\right\} \tag{5}$$

We also express $\mathbf{A}$ as the adjacency matrix representing the mesh connectivity as follows:

$$\mathbf{A}_{ij} = \begin{cases} 1 & \text{if } \; (v_i, v_j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

In fact, the resulting matrix $\left(\mathbf{I} - \left(\mathbf{D}^{-1} \cdot \mathbf{A}\right)\right)$ is no other than the Laplacian matrix of the input mesh. The square matrix $(\mathbf{I} - diag\,(\mathbf{B}))$ has the effect of removing all connectivity information everywhere into the row associated with a vertex belonging to the Dirichlet boundary condition, while the square matrix $(diag\,(\mathbf{B}))$ specifies the strong constraint for each vertex of the brushed boundaries. The combination of both terms injects the boundary condition strongly into the system. This scheme eliminates parasitizing information that makes the system previously weakly formulated (see Fig. 3).

Our new wrapped formulation strongly imposes boundaries into the traditional Laplacian matrix, using this matrix as a natural support to inject the boundary condition inside. This mathematical wrapping ensures that all values will remain unchanged at boundaries. We called the resulting left-hand side matrix of our system the *Dirichlet-Laplacian Matrix* since it forms the desired square matrix with boundary condition included in the system. This injection defines boundary condition as strong constrained directly into the Laplacian matrix without using the standard splitting method employed by almost all previous works. The discrete Poisson equation yields a linear system that can be solved numerically by linear least square minimization. We employ sparse matrices to obtain an efficiency packing of matrix coefficients. Therefore, we solve the system in the least-squares sense using a conjugate gradient black-box solver offered by `OpenNL` library.

Poisson Mesh Isolines Sampling. Isolines are defined as a collection of segments connecting surface points having the same isovalue. Since the field is smooth, partial segments look connected as isolines. We uniformly sample isolines over the estimated harmonic field. Since isolines are dependent on the mesh geometry, the resulting Poisson-Dirichlet isolines are shape-aware. As shown in Figure 4, regular sampling of isolines can also be bounded inside a sliding window, using the color ramp as a selective interface, making isolines sampling more intuitive than earlier techniques.

4 Results and Discussion

In the following section, we present numerical results of our strong formulation of the Dirichlet boundary condition. We also discuss the benefits and limitations of our technique.

Results and Timing. Our prototype has been implemented using C/C++ and the `OpenNL` library. We apply our algorithm to complex triangulated shapes using an Intel Core workstation with 4Gb of memory and Geforce GTX 660M. Figure 1 and 2 show our algorithm applied to the Dragon and Fertility models. Table 1 gives statistics on the computational times of our approach. For instance, for the Stanford Dragon model with 28190 faces and 14101 vertices the whole process takes 7.73 seconds to sample 13 isolines (see Figure 1). The measured overall timing includes the mesh connectivity computation, the Laplacian calculation, the solving process for the linear system (on CPU), as well as the isolines sampling. In essence, the computational time depends on the number of vertices linearly, whatever the configuration of the boundary condition. Additional results can be found in our supplementary video.

Benefits and Limitations. Our approach is advantageous since no approximation has to be performed at the boundaries. Contrary to the splitting method employed in previous work, Dirichlet boundary conditions are exactly satisfied in our work. We take advantage of the system linearity to address the initial/boundary conditions. The benefit is that system is maintained as squared, and the boundary condition is accurately imposed leading to potential efficient computation [Bolz et al. 2003]. Thus, the resulting system can be solved efficiently with a direct linear solver, without the expensive computation of the so-called normal equations matrix. Estimated isolines retain important properties such as the convex hull property and partition of unity while providing a continuous mapping from a one-dimensional color space to the 3D model space. Even if the Laplace equation is the steady-state heat equation, an inherent limitation is that the Dirichlet problem potentially offers a distribution of heat, but it does not measure heat. Finally, we observed that our numerical solution is robust to topological changes and local geometrical changes.

Figure 4: ***Competitive-boundaries for surface isolines.*** *A color-coded harmonic field is obtained by solving the boundary-aware Poisson equation (**left-hand side**). Then, isolines (superimposed in red lines) are extracted by sampling the harmonic field inside a bounded interval (**middle**). We show a close-up view of the bounded sampling restricting the isolines selection to lie inside a range of the color map. The color-code corresponds to the isovalue* $\in [0, 1]$ *of the harmonic field (**right-hand side**).*

Statistics	Dragon	Fertility
Vertex number	14101	4994
Face number	28190	10000
Mesh Connectivity	0.71 s.	96 ms.
Harmonic Solver	6.87 s.	0.82 s.
10 Isolines Sampling	15 ms.	4.4 ms.
13 Isolines Sampling	24 ms.	5.6 ms.

Table 1: ***Computational times.*** *We detail here the statistics for the Dragon (Fig. 1) and Fertility model (Fig. 2).*

5 Conclusions

This paper proposes a useful scheme to impose Dirichlet boundary conditions as hard constraints for the Poisson equation without any approximation while finding a harmonic function. A large body of recent work has shown that Poisson-based isolines are of great interest for shape analysis. Still, defining suitable numerical boundaries with a strong formulation is a hard problem for complex boundary conditions types. Boundary conditions management is hard to setup when dealing with partial differential equations. The key to our approach is the ability to inject strong boundary conditions into the Laplacian matrix while keeping the final matrix in square form. Also, our coloring-guided isolines extraction technique provides an automatic and intuitive selection for a spatially coherent and isovalue-bounded set of isolines, leading to an aesthetic visualization at high numerical performance. To the best of our knowledge, our method is the first to provide an injective solution to the problem of Dirichlet boundary condition for Poisson Equation. Building upon a Laplacian-type system and variational strategies, the strong boundary treatment makes our isoline-based system unique. A potential issue is to adapt our system for Cauchy or Robin boundary condition. A promising future work is to extend our framework towards polyharmonic isolines.

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